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Resonant harmonic oscillators and eigenvalue multiplicity

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Abstract. Explicit formulae are worked out for the eigenvalue multiplicity of a system of n independent quantum harmonic oscillators in the general case of $1 \leqslant s \leqslant n-1$ resonance relations among the frequencies $\omega_1, \ldots, \omega_n$. As a particular case we prove that, even though the quantum numbers are always less than the degrees of freedom, the eigenvalues are, in general, intrinsically degenerate only in the completely resonant case $s = n-1$.

1. Introduction

Consider a system of n independent quantum oscillators with frequencies $\omega_1, \ldots, \omega_n$. Setting $\hbar = 1$, the eigenvalues are

$$\lambda_{p_1,\ldots,p_n} = \omega_1(p_1 + \tfrac{1}{2}) + \cdots + \omega_n(p_n + \tfrac{1}{2}) \qquad p_k = 0, 1, \ldots \quad k = 1, \ldots, n\,. \tag{1.1}$$

It is well known (see e.g. [Bo section 2.15]) that if there are resonance relations, i.e. rational relations among the frequencies, the quantum conditions (equivalently, the quantum numbers) are less than the number of degrees of freedom and hence the eigenvalues are expected to be intrinsically degenerate; namely with multiplicity greater than one except for the ground state. In the simplest (called completely) resonant case, which takes place when all frequencies are equal, $\omega_1 = \cdots = \omega_n = \omega$, we have $\lambda_{p_1,\ldots,p_n} = (N + \frac{1}{2}n)\omega$, where $N = p_1 + \cdots + p_n$, and the multiplicity is (see e.g. [Me]) $M(N, n) = (N+n-1)!/N!(n-1)!$; namely, $M(N, n)$ is precisely the number of eigenvalues $\lambda_{p_1,\ldots,p_n}$ such that, for any $N \in \mathbb{N}$, $\lambda_{p_1,\ldots,p_n} = (N + \frac{1}{2}n)\omega$.

Despite its spontaneous nature, however, there is to our knowledge no general reply to the following natural and apparently simple question: what is the multiplicity of the eigenvalues in the most general resonant case, i.e. of the occurrence of $s : 1 \leqslant s \leqslant n-1$ resonance relations among the frequencies; namely, when there exist integer numbers $\nu_{ij} : i = 1, \ldots, s;\ j = 1, \ldots, n$ with $\nu_{ij} \neq 0$ for at least two pairs of indices such that

$$\left\{\begin{array}{lclcl} \nu_{11}\omega_1 & +\ldots+ & \nu_{1n}\omega_n & = & 0 \\ \nu_{21}\omega_1 & +\ldots+ & \nu_{2n}\omega_n & = & 0 \\ \vdots & +\ldots+ & \vdots & & \vdots \\ \nu_{s1}\omega_1 & +\ldots+ & \nu_{sn}\omega_n & = & 0\,. \end{array}\right. \tag{1.2}$$

There are, moreover, at least two recent developments which require this multiplicity computation as a preliminary step: the extension to the resonant case of the statistics of eigenvalues (1.1) as a function of the arithmetic properties of the frequencies determined

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by Bleher [Bl1, Bl2], and the extension to the intrinsically resonant case of the exact quantization of canonical perturbation theory [Gr, GP, DEGH].

The purpose of this paper is therefore to work out an expression for the above multiplicity denoted by $M_n^s(\nu_{ij}, p) : p = (p_1, \ldots, p_n), \nu_{ij} = (\nu_{i1}, \ldots, \nu_{in}) : i = 1, \ldots, s$. The result is admittedly rather involved in the most general case, but explicit and simple in the two possibly most important particular cases, namely $s = n - 1$ and $s = 1$, the maximum and minimum number of resonance relations allowed, respectively. The reason preventing the existence of such a result in the literature could indeed be that the computation of the multiplicity, trivial for $n = 2$ or for $n = 3, s = 2$ (for example, $M_2^1(\nu_{11}, \nu_{12}, p) = [(p_1/\nu_{11}) + (p_2/\nu_{12})]$ where $[x]$ denotes the integer part of x) becomes rapidly very cumbersome when n and s increase.

This computation of the multiplicity is in fact related to a well known problem in number theory (a slight variant of it is known as the postage-stamp problem, see e.g. [RS, Se1, Se2]); namely, the counting of the non-negative solutions of a linear diophantine system (for this general topic see e.g. [Ca]). Hence this is the way in which the solution will be obtained, and this counting procedure highlights an important difference from the completely resonant (i.e. equal frequencies) case: if $s < n-1$, the frequencies can always be selected in such a way as to generate sequences of eigenvalues of *constant multiplicity* (in particular, 1) diverging to infinity. In other words, despite the fact that for a system of n independent oscillators the quantum conditions are less than the number of degrees of freedom no matter how the frequencies are chosen *provided they admit at least one resonance relation*, if $s < n - 1$ the eigenvalues are actually intrinsically degenerate only for special values of the frequencies: otherwise, there can be simple eigenvalues (the degeneracy of the spectrum being therefore accidental) and the multiplicity of any eigenvalue tends to infinity as the 'quantum numbers' increase to infinity if and only if the rs relations are exactly $n - 1$.

To formulate our results let us first specify some notational conventions. We consider from now n independent quantum harmonic oscillators with the zero-point energy subtracted, i.e. the operators in $L^2(\mathbb{R}^n)$ defined by the maximal action of the following differential expression:

$$H = -\tfrac{1}{2}\Delta + \tfrac{1}{2}\left(\omega_1^2 q_1^2 + \cdots + \omega_n^2 q_n^2\right) - \tfrac{1}{2}(\omega_1 + \ldots + \omega_n)$$

so that the eigenvalues (1.1) become

$$\lambda_{p_1,\ldots,p_n} = \omega_1 p_1 + \cdots + \omega_n p_n \equiv \langle \omega, p \rangle \qquad p_k = 0, 1, \ldots \quad k = 1, \ldots, n \tag{1.3}$$

with $p = (p_1, \ldots, p_n)$, $\omega = (\omega_1, \ldots, \omega_n)$. We repeat that the frequencies $\omega_1, \ldots, \omega_n$ admit $1 \leqslant s \leqslant n$ resonance relations if there exists an $s \times n$ matrix A of rank s with integer elements, which without loss of generality can always be assumed in row-reduced form, such that

$$A\omega = 0\,. \tag{1.4}$$

Explicitly, setting

$$A = \begin{pmatrix} \nu_{1,1} & \ldots & \nu_{1,n-s} & -\nu_{1,n-s+1} & 0 & \ldots & 0 \\ \nu_{2,1} & \ldots & \nu_{2,n-s} & 0 & -\nu_{2,n-s+2} & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & 0 & \ddots & \vdots \\ \nu_{s,1} & \ldots & \nu_{s,n-s} & 0 & 0 & \ldots & -\nu_{s,n} \end{pmatrix} \tag{1.5}$$

equation (1.4) becomes

$$\begin{cases} \nu_{1,n-s+1}\omega_{n-s+1} & = & \sum_{j=1}^{n-s} \nu_{1,j}\omega_j \\ \vdots & \vdots & \vdots \\ \nu_{s,n}\omega_n & = & \sum_{j=1}^{n-s} \nu_{s,j}\omega_j \,. \end{cases} \tag{1.6}$$

Now setting

$$\alpha = \prod_{j=1}^{s} \nu_{j,n-s+j} \qquad q_{ij} = \nu_{ji} \prod_{k\neq j=1}^{s} \nu_{k,n-s+k} \tag{1.7}$$

we can define exactly $n-s$ 'principal quantum numbers' $N_i^{(s)}$ in the following way:

$$N_i^{(s)} = p_i\alpha + \sum_{j=1}^{s} q_{ij}p_{n-s+j} \qquad i = 1,\ldots,n-s\,. \tag{1.8}$$

In this way one can write

$$\lambda_{p_1,\ldots,p_n} = \frac{1}{\alpha}\sum_{i=1}^{n-s} N_i^{(s)}\omega_i \tag{1.9}$$

and hence we can conclude that *the multiplicity of the eigenvalue* $\lambda_{p_1,\ldots,p_n}$ *is given, for any fixed integer value of* $N_i^{(s)}: i = 1,\ldots,n-s$, *by the number of non-negative integer solutions* $p_1,\ldots,p_n$ *of (1.8).*

Concerning this statement, we remark on the following.

(i) Formula (1.8) does not define the quantum numbers uniquely, except for $s = n-1$. In this case we recover the usual notion of principal quantum number, since (1.8) yields

$$N = p_1\alpha + \sum_{j=1}^{n-1} q_{1j}p_{j+1} \tag{1.10}$$

where, as we shall see, both α and q_{1j}: $j = 1,\ldots,n-1$ are positive.
Actually in the simplest possible case, namely $s = n-1$, $\omega_1 = \cdots = \omega_n$ which yields $\alpha = 1, q_{ij} \equiv q = 1$, this becomes $N = p_1 + p_2 + \cdots + p_n$, the principal quantum number of the n-dimensional, equal-frequencies oscillator.

(ii) In the general case of arbitrary s, however, the $n-s$ 'principal quantum numbers' (1.8) need not be positive definite. However, the solutions $p_1,\ldots,p_n$ of (1.8) must always be non-negative and this constraint allows us to estimate the multiplicity, making use of the results of [BT].

(iii) The form of (1.9) allows us to make more precise the above remarks on intrinsic degeneracy and on constant multiplicity. The quantum numbers are less than the degrees of freedom since $s \geqslant 1$, and, moreover, we expect the multiplicity to always tend to infinity when at least $\sum_{i=1}^{n-s} N^{(s)} \to \infty$. We will see below, on the contrary, that if $s \leqslant n-2$ there exist (always for $s = 1$, and for suitable choice of the frequencies for $1 < s < n-1$) sequences of eigenvalues of constant multiplicity (in particular, 1) tending to infinity. Two relevant examples are explicitly worked out in the remark after proposition 2.2. Therefore for a general choice of the resonant frequencies the spectrum is actually intrinsically degenerate, with multiplicity tending to infinity as the quantum numbers diverge, if and only if $s = n-1$, namely in the maximally resonant case. Moreover, if there is only one resonance relation the spectrum is always accidentally degenerate.

(iv) There is no loss of generality in assuming that the $s \times (n-s)$ matrix formed by the first $n-s$ columns of (1.5) does not admit any block reduction. Indeed the block reducibility corresponds to a set of independent resonance relations among independent degrees of freedom. The total multiplicty would therefore be the product of the multiplicities generated by any single resonance relation.

2. Statement of the results

We can now begin to state the results. As already mentioned, for the sake of simplicity we prefer to describe separately the two particular cases, namely $s = n-1$ and $s = 1$, the maximum and minimum number of resonance relations allowed, respectively. For $s = n-1$ we have just the quantum number N given by (1.10). Denoting once more $M_n^s(p_1, \ldots, p_n)$ the multiplicity in the case of n degrees of freedom and s resonance relations, the result is given by the following.

Proposition 2.1. Let $s = n-1$, and $\alpha, q_{1,j}, N$ be defined as in (1.7) and (1.10), respectively. Then the multiplicity $M_n^{n-1}(\alpha, q_{1,1}, \ldots, q_{1,n-1}; N)$ of the eigenvalue $\lambda_{p_1,\ldots,p_n} = N\omega_1/\alpha$ is given by

$$M_n^{n-1}(\alpha, q_{1,1}, \cdots, q_{1,n-1}; N) = \sum_{k_3=0}^{K_3} \cdots \sum_{k_n=0}^{K_n} M_2^1\Big(\alpha, q_{1,1}; N - \sum_{j=3}^{N} k_j q_{1,j+1}\Big) \tag{2.1}$$

where $[x]$ denotes the integer part of x, $K_j = [N/q_{1,j-1}]$, $D_2 = \gcd(q_{11}, \alpha)$ and

$$M_2^1(\alpha, q_{1,1}; N) = \begin{cases} D_2\left[\dfrac{N}{\alpha q_{1,1}}\right] + B & \text{if} \quad N \geqslant 0 \ \text{ and } \ D_2|N \\ 0 & \text{otherwise} \end{cases} \tag{2.2}$$

for some $B \in \mathbb{Z}$, $0 \leqslant B \leqslant D_2$. Here and in what follows the symbol $D_2|N$ indicates that D_2 divides N.

Remarks

(i) A constructive formula for computing the integer B in (2.2) will be given below in the proof of lemma 3.1.

(ii) Consider once more the simplest case $\omega_1 = \omega_2$. Then we have $\alpha = q_{1,1} = 1, N = p_1 + p_2, D_2 = 1, B = 1$ and hence $m_2^1 = N + 1$. For n degrees of freedom and frequencies $\omega_1 = \cdots = \omega_n$ the expression (2.1) gives the well known result

$$M_n^{n-1}(1, \ldots, 1; N) = \frac{(N+n-1)!}{N!(n-1)!}. \tag{2.3}$$

In fact one has (by induction),

$$\begin{aligned} M_n^{n-1}(\underbrace{1, \ldots, 1}_{n}; N) &= \sum_{k_3=0}^{N} \cdots \sum_{k_n=0}^{N} M_2^1\Big(1, 1; N - \sum_{j=3}^{N} k_j\Big) \\ &= \sum_{k_3=0}^{N} M_{n-1}^{n-2}(\underbrace{1, \ldots, 1}_{n-1}; N - k_3) \\ &= \sum_{k_3=0}^{N} \frac{(N+n-2-k_3)!}{(N-k_3)!(n-2)!} = \frac{(N+n-1)!}{N!(n-1)!}. \end{aligned}$$

Let us now state the result for the single resonance case, i.e. $s = 1$, where there are $n-1$ quantum numbers $N_1, \ldots, N_{n-1}$.
Denote as above $D_n = \gcd(\alpha; q_{11}, \ldots, q_{n-1,1})$.
Let $0 \leqslant y_n^k < \alpha : k = 1, \ldots, D_n$ be all solutions of the linear congruence system

$$\left.\begin{array}{ccc} q_{1,1}x_n & \equiv & N_1 \\ \vdots & & \vdots \\ q_{n-1,1}x_n & \equiv & N_{n-1} \end{array}\right\} \pmod{\alpha} \tag{2.4}$$

and y_i^k the correspondent solutions of linear system

$$\left\{\begin{array}{ccccc} q_{1,1}x_n & + & \alpha x_1 & = & N_1 \\ \vdots & & \vdots & & \vdots \\ q_{n-1,1}x_n & + & \alpha x_{n-1} & = & N_{n-1}\,. \end{array}\right. \tag{2.5}$$

Then we have the following.

Proposition 2.2. The multiplicity $M_n^1(\alpha, q_{1,1}, \ldots, q_{n-1,1}; N_1, \ldots, N_{n-1})$ has the following expression:

$$M_n^1(\alpha, q_{1,1}, \ldots, q_{n-1,1}; N_1, \ldots, N_{n-1}) = \sum_{k=1}^{D_n} \max(0, [L_k - M_k] + 1) \tag{2.6}$$

where

$$\begin{aligned} L_k &= \min_{r+1 \leqslant i \leqslant n-1} \left[\frac{y_i^k}{q_{i,1}}\right] \qquad k = 1, \ldots, D_n \\ M_k &= \max\left(0, \left(\frac{1}{\alpha}\left(\max_{j \leqslant r}\left(0, \frac{N_j}{q_{j,1}}\right) - y_n^k\right)\right)\right). \end{aligned} \tag{2.7}$$

Here $0 \leqslant r \leqslant n-2$ is the unique non-negative integer such that $q_{1,1}, \ldots, q_{r,1} < 0$; $q_{r+1,1}, \ldots, q_{n-1,1} > 0$, and, $\forall i = r+1, \ldots, n-1$

$$\left[\frac{y_i^k}{q_{i,1}}\right] = \left[\frac{N_i}{q_{i,1}}\right] \quad \text{or} \quad \left[\frac{N_i}{q_{i,1}}\right] - 1\,. \tag{2.8}$$

Remarks

(i) The existence of $r : 0 \leqslant r \leqslant n-2$ such that $q_{1,1}, \ldots, q_{r,1} < 0; q_{r+1,1}, \ldots, q_{n-1,1} > 0$ is easily proved. Let us first recall that if there is just one resonance relation we have by definition $-\nu_{1,j} = q_{j,1}, \quad j = 1, \ldots, n-1$; $\nu_{1,n} = \alpha$ and therefore we may write $\omega_n = (1/\alpha)\sum_{j=1}^{n-1} \nu_{1,j}\omega_j$. Choosing $\alpha > 0$ there is at least one index i for which $q_{i,1} > 0$. Otherwise

$$0 < \omega_n = \sum_{j=1}^{n-1} \frac{\nu_{1,j}}{\alpha}\omega_j = \sum_{j=1}^{n-1} \frac{q_{j,1}}{\alpha}\omega_j < 0\,.$$

Therefore in what follows we can assume the existence of $0 \leqslant r \leqslant n-2$ such that $\nu_{1,1}, \ldots, \nu_{1,r} < 0$; $\nu_{1,r+1}, \ldots, \nu_{1,n-1} > 0$.

(ii) The behaviour of $M_n^1(\cdot)$ is very irregular. As already remarked we can always find sequences of eigenvalues of constant multiplicity tending to infinity, in particular sequences of simple eigenvalues. The same thing happens for $M_n^s(\cdot)$ $1 < s < n-1$, but only for suitable choice of frequencies. Let us present two simple examples of these situations.

Example 1. Consider for $s = 1$ the following sequence of 'principal quantum numbers' $N^{(h)} = (N_1^{(h)}, \dots, N_{n-1}^{(h)})$, where

$$\begin{aligned} N_i^{(h)} &= h\alpha \qquad i = 1, \dots, n-2 \quad \forall h \in \mathbb{N} \\ N_{n-1}^{(h)} &= 0 \qquad \forall h \in \mathbb{N}. \end{aligned} \tag{2.9}$$

Let us compute $M_n^1(\alpha, q_{1,1}, \dots, q_{n-1,1}; N_1^{(h)}, \dots, N_{n-1}^{(h)})$: this function is given ($\forall h \in \mathbb{N}$) by the number of the non-negative integer solutions of the system

$$\left\{ \begin{array}{lclcl} q_{1,1}x_n & + & \alpha x_1 & = & h\alpha \\ \vdots & & \vdots & & \vdots \\ q_{n-2,1}x_n & + & \alpha x_{n-2} & = & h\alpha \\ q_{n-1,1}x_n & + & \alpha x_{n-1} & = & 0. \end{array} \right. \tag{2.10}$$

Since α and $q_{n-1,1}$ are both positive, it is clear that $\forall h \in \mathbb{N}$ this system only admits the solution

$$\begin{array}{lcll} y_n & = & 0 & \Rightarrow \\ y_1 & = & h & \\ \vdots & & \vdots & \\ y_{n-2} & = & h & \\ y_{n-1} & = & 0. & \end{array} \tag{2.11}$$

Therefore $M_n^1(\alpha, q_{1,1}, \dots, q_{n-1,1}; N_1^{(h)}, \dots, N_{n-1}^{(h)}) = 1 \quad \forall h \in \mathbb{N}$, while it is clear that $\sum_{i=1}^{n-1} N_i^{(h)} \to +\infty$ and $\lambda_{N_1^{(h)}, \dots, N_{n-1}^{(h)}} = (1/\alpha) \sum_{i=1}^{n-1} N_i^{(h)} \omega_i \to +\infty$.

Example 2. In the general case $s > 1$ consider all frequencies $\omega_1, \dots, \omega_n$ for which the resonance relations give $q_{n-s,j} > 0 \quad \forall j = 1, \dots, s$ (see (1.7) for the definition of $q_{i,j}$). Consider now the following sequence of 'principal quantum numbers' $N^{(h)} = (N_1^{(h)}, \dots, N_{n-s}^{(h)})$ where

$$\begin{aligned} N_i^{(h)} &= h\alpha \qquad i = 1, \dots, n-s-1 \quad \forall h \in \mathbb{N} \\ N_{n-s}^{(h)} &= 0 \qquad \forall h \in \mathbb{N}. \end{aligned} \tag{2.12}$$

The multiplicity $M_n^s(\cdot)$ of the correspondent eigenvalues is given by the number of non-negative integer solutions of the system

$$\left\{ \begin{array}{ccccccccc} q_{1,1}x_{n-s+1} & + \cdots + & q_{1,s}x_n & + & \alpha x_1 & = & h\alpha \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ q_{n-s-1,1}x_{n-s+1} & + \cdots + & q_{n-s-1,s}x_n & + & \alpha x_{n-s-1} & = & h\alpha \\ q_{n-s,1}x_{n-s+1} & + \cdots + & q_{n-s,s}x_n & + & \alpha x_{n-s} & = & 0. \end{array} \right. \tag{2.13}$$

$\forall h \in \mathbb{N}$ this system admits only the solution

$$y_{n-s} = \cdots = y_n = 0 \quad \Rightarrow \quad y_1 = \cdots = y_{n-s-1} = h. \tag{2.14}$$

Therefore

$$M_n^s(\alpha, q_{i,j} | i = 1, \dots, n-s; j = 1, \dots, s; N_1^{(h)}, \dots, N_{n-1}^{(h)}) = 1 \quad \forall h \in \mathbb{N}$$

while it is clear that $\lambda_{N_1^{(h)}, \dots, N_{n-s}^{(h)}} = (1/\alpha) \sum_{i=1}^{n-s} N_i^{(h)} \omega_i \to +\infty$.

We are now ready to state the result valid for all the remaining cases, namely $1 < s < n-2$. Denoting

$$\begin{aligned} H &= (h_{n-s+2}, \dots, h_n) \in \mathbb{Z}^{s-1} \\ R_i^H &= N_i - \sum_{j=2}^{s} h_{n-s+j} q_{i,j} \qquad i = 1, \dots, n-s \end{aligned} \tag{2.15}$$

and introducing the matrices

$$\mathcal{R}_1 = \begin{pmatrix} q_{1,1} \\ \vdots \\ q_{n-s,1} \end{pmatrix} \qquad \mathcal{R}_2^H = \begin{pmatrix} q_{1,1} & R_1^H \\ \vdots & \vdots \\ q_{n-s,1} & R_{n-s}^H \end{pmatrix} \tag{2.16}$$

and the elementary divisors $e_1; \epsilon_1^H, \epsilon_2^H$ of the matrices $\mathcal{R}_1$ and $\mathcal{R}_2^H$, respectively (recall that the first elementary divisor ϵ_1 of a matrix is the gcd among its elements and the second elementary divisor ϵ_2 is Δ_2/ϵ_1, where Δ_2 denotes the gcd among the absolute values of all rank-2 minors), we have:

Proposition 2.3. The multiplicity

$$M_n^s(\alpha, q_{i,j}|_{i=1,\dots,n-s; j=1,\dots,s}; N_1, \dots, N_{n-s})$$

can be computed in the following way:

$$M_n^s(\cdot) = \sum_{\substack{H \in \mathbb{Z}^{s-1} \\ 0 \leqslant h_i \leqslant K}} M_{n-s+1}^1(\alpha, q_{1,1}, \dots, q_{n-s,1}; R_1^H, \dots, R_{n-s}^H) \tag{2.17}$$

where M_{n-s+1}^1 is the multiplicity function of a system of $n-s+1$ independent oscillators admitting one resonance relation computed in proposition 2.2, extended to zero whenever *at least one of the following conditions:*

$$\begin{aligned} &\text{(i)} \quad \gcd(e_1, \alpha) = \gcd(\epsilon_1^H, \alpha) \\ &\text{(ii)} \quad \epsilon_2^H \equiv 0 \qquad (\text{mod } \alpha) \\ &\text{(iii)} \quad R_i^H \geqslant 0 \qquad \forall i \quad \text{such that } q_{i,1} \geqslant 0 \end{aligned} \tag{2.18}$$

is not fulfilled; $K = \max_p |\Delta_p|$, where $\Delta_p : p = 1, \dots, (n+1)!/(n-s)!(s+1)!$ are the minors of order $n-s$ of the $(n-s) \times (n+1)$ matrix,

$$\begin{pmatrix} \alpha & 0 & \dots & 0 & q_{1,1} & \dots & q_{1,s} & N_1 \\ 0 & \alpha & 0 & 0 & q_{2,1} & \dots & q_{2,s} & N_2 \\ \vdots & 0 & \ddots & \vdots & \vdots & & \vdots & \vdots \\ 0 & \dots & 0 & \alpha & q_{n-s,1} & \dots & q_{n-s,s} & N_{n-s} \end{pmatrix}. \tag{2.19}$$

Example. Let us compute the multiplicity of the eigenvalues in two simple cases.

(i) Let $n = 3$ and consider the frequencies

$$\omega_1 = 1 \qquad \omega_2 = e \qquad \omega_3 = 1 + e$$

yielding the resonance relation $\omega_1 + \omega_2 - \omega_3 = 0$. Let N_1, N_2 be the 'quantum numbers' defined in (1.8). It follows that: $\alpha = 1 = q_{1,1} = q_{2,1}$, $D_3 = 1$ and the solutions y_i of systems (2.4), (2.5) are trivially

$$y_3 = 0 \qquad y_1 = N_1 \qquad y_2 = N_2\,.$$

Therefore the multiplicity $M_3^1(1, 1, 1; N_1, N_2)$ is

$$M_3^1(1, 1, 1; N_1, N_2) = \max(0, [L - M] + 1)$$

where $L = \min(N_1, N_2)$ $M = \max(0, -1) = 0$

$$\Rightarrow \quad M_3^1(1, 1, 1; N_1, N_2) = (\min(N_1, N_2)) + 1\,.$$

(ii) Let $n = 4$ and consider the frequencies

$$\omega_1 = 1 \qquad \omega_2 = e \qquad \omega_3 = 1 + e \qquad \omega_4 = 1 - e$$

yielding the two resonance relations

$$\begin{cases} \omega_1 + \omega_2 - \omega_3 = 0 \\ \omega_1 - \omega_2 - \omega_4 = 0\,. \end{cases}$$

The coefficients defined in (1.7) are

$$\alpha = 1 \qquad q_{1,1} = 1 \qquad q_{1,2} = 1 \qquad q_{2,1} = 1 \qquad q_{2,2} = -1\,.$$

Let $N_1,\ N_2$ be the 'quantum numbers' defined in (1.8); let $H = h_4 \in \mathbb{Z}$ and

$$R_1^H = N_1 - h_4 \qquad R_2^H = N_2 + h_4\,.$$

The matrix (2.19) is

$$\begin{pmatrix} 1 & 0 & 1 & 1 & N_1 \\ 0 & 1 & 1 & -1 & N_2 \end{pmatrix}$$

and it follows that (we consider only the non-trivial cases $N_i \geqslant 1,\ i = 1, 2$)

$$K = N_1 + N_2\,.$$

Therefore, by (2.18)(iii) and example (i) the multiplicity $M_4^2(1, 1, 1, 1, -1; N_1, N_2)$ is

$$\begin{aligned} M_4^2(\cdot) &= \sum_{0 \leqslant h_4 \leqslant N_1 + N_2} M_3^1(1, 1, 1; N_1 - h_4, N_2 + h_4) \\ &= \sum_{0 \leqslant h_4 \leqslant N_1} M_3^1(1, 1, 1; N_1 - h_4, N_2 + h_4) \\ &= \sum_{0 \leqslant h_4 \leqslant N_1} (\min(N_1 - h_4, N_2 + h_4) + 1)\,. \end{aligned}$$

3. Proof of the results

The first step towards the proof of proposition 2.1 is obviously represented by the computation of multiplicity function $M_2^1(\alpha, q_{1,1}; N)$, i.e. the number of the non-negative integer solutions of the linear diophantine equation

$$\alpha x_1 + q_{1,1} x_2 = N \tag{3.1}$$

where $\alpha \in \mathbb{N}$, $q_{1,1} \in \mathbb{N}$ but we have to take $N \in \mathbb{Z}$.

Lemma 3.1. Let $D_2 = \gcd(\alpha, q_{1,1})$ and let (y_1^h, y_2^h) : $h = 1, \ldots, D_2$ be the D_2 distinct solutions of (3.1) such that $q_{1,1} y_2^h \equiv N (\mathrm{mod}\ \alpha)$. Then

$$M_2^1(\alpha, q_{1,1}; N) = \begin{cases} D_2 + \sum_{h=1}^{D_2} \left[y_1^h / q_{1,1} \right] & \text{if} \quad N \geqslant 0 \quad \text{and} \quad D_2 | N \\ 0 & \text{otherwise}\,. \end{cases} \tag{3.2}$$

Moreover, for $h = 1, \ldots, D_2$, the function $\left[y_1^h / q_{1,1} \right]$ assumes just two values:

$$\left[\frac{y_1^h}{q_{1,1}} \right] = \left[\frac{N}{q_{1,1}\alpha} \right] \qquad \text{or} \qquad \left[\frac{N}{q_{1,1}\alpha} \right] - 1 \tag{3.3}$$

and

$$M_2^1(\alpha, q_{1,1}; N) = \begin{cases} D_2\left[\dfrac{N}{q_{1,1}\alpha}\right] + B & \text{if } N \geqslant 0 \text{ and } D_2|N \\ 0 & \text{otherwise} \end{cases} \tag{3.4}$$

where $0 \leqslant B \leqslant D_2$.

Proof. Remark that, obviously, $M_2^1(\cdot) = 0$ if $N < 0$ and/or D_2 does not divide N. Otherwise, consider first the case $D_2 = 1$, and let (y_1, y_2) be the unique solution of (3.1) such that $q_{1,1}y_2 \equiv N(\text{mod } \alpha)$. Since the most general solution is given by

$$(y_1, y_2) + (-kq_{1,1}, k\alpha) \qquad k \in \mathbb{Z} \tag{3.5}$$

it follows that y_1 is the largest value which x_1 may assume if x_2 is required to be non-negative. Moreover it is easily seen that

$$\frac{N}{\alpha} - q_{1,1} < y_1 \leqslant \frac{N}{\alpha}. \tag{3.6}$$

Hence by (3.5) all non-negative solutions of (3.1) have the form

$$(y_1, y_2) + (-kq_{1,1}, k\alpha) \qquad 0 \leqslant k \leqslant \left[\frac{y_1}{q_{1,1}}\right]. \tag{3.7}$$

It follows immediately

$$M_2^1(\alpha, q_{1,1}; N) = \left[\frac{y_1}{q_{1,1}}\right] + 1 \tag{3.8}$$

and, by (3.6), the validity of the estimates

$$\left[\frac{N}{\alpha q_{1,1}}\right] - 1 \leqslant \left[\frac{y_1}{q_{1,1}}\right] \leqslant \left[\frac{N}{\alpha q_{1,1}}\right]. \tag{3.9}$$

If now $D_2 > 1$, the same argument applies to any single one of the D_2 distinct solutions (y_1^h, y_2^h): $h = 1, \ldots, D_2$ of (3.1) such that $q_{1,1}y_2^h \equiv N(\text{mod } \alpha)$. This yields all non-negative solutions of (3.1) under the form

$$(y_1^h, y_2^h) + (-k(h)q_{1,1}, k(h)\alpha) \qquad h = 1, \ldots, D_2 \quad 0 \leqslant k(h) \leqslant \left[\frac{y_1^h}{q_{1,1}}\right]. \tag{3.10}$$

Hence (3.9) holds for any $h : 1 \leqslant h \leqslant D_2$, which proves (3.3), and summing over h we get (3.4). This concludes the proof of lemma 3.1.

Proof of proposition 2.1. The multiplicity of the eigenvalue $\lambda_{p_1,\ldots,p_n}$ is the number of non-negative solutions of the linear diophantine equation

$$\sum_{i=1}^{n-1} q_{1,i}x_{i+1} + x_1\alpha = N. \tag{3.11}$$

For $n = 2, s = 1$, equation (3.11) reduces to (3.1) because the resonance relation can be written under the form $\omega_2 = (q_{1,1}/\alpha)\omega_1$. Therefore we can take from now on $n > 2$. Given $n - 3$ non-negative integers $k_j : j = 3, \ldots, n$ we have to find the number of solutions of the equations parametrized by k_j:

$$\alpha x_1 + q_{1,1}x_2 = N - \sum_{j=3}^{n} k_j q_{1,j-1} \qquad k_j \in \mathbb{Z};\ j = 3, \ldots, n. \tag{3.12}$$

It is indeed plain that the $n$ple $(y_1, y_2; k_3, \ldots, k_n)$ is a non-negative solution of (3.11) if and only if (y_1, y_2) is a non-negative solution of (3.12). By lemma 3.1 the number of non-negative solutions of (3.12) is

$$M_2^1\left(\alpha, q_{1,1}; N - \sum_{j=3}^{n} k_j q_{1,j-1}\right). \tag{3.13}$$

Therefore one formally has

$$M_n^{n-1}(\cdot) = \sum_{k_3 \in \mathbb{Z}_+} \cdots \sum_{k_n \in \mathbb{Z}_+} M_2^1\left(\alpha, q_{1,1}; N - \sum_{j=3}^{n} k_j q_{1,j-1}\right). \tag{3.14}$$

However, by the former lemma the function $M_2^1(\cdot)$ vanishes for $N - \sum_{j=3}^{n} k_j q_{1,j-1} < 0$. Hence, for any $3 \leqslant j \leqslant n$, the upper limit of the sum over k_j is $[N/q_{1,j-1}]$ which proves the assertion. This concludes the proof of proposition 2.1.

Let us turn now to the proof of proposition 2.2. Let us first recall that if there is just one resonance relation we have by definition $-\nu_{1,j} = q_{j,1}$ $j = 1, \ldots, n-1$; $\nu_{1,n} = \alpha$ and we may write $\omega_n = (1/\alpha)\sum_{j=1}^{n-1} \nu_{1,j}\omega_j$. Moreover the 'quantum numbers' defined in (1.8) read

$$N_i = p_i\alpha + q_{i,1}p_n \qquad i = 1, \ldots, n-1 \tag{3.15}$$

so that the multiplicity $M_n^1(\alpha, q_{1,1}, \ldots, q_{n-1,1}; N_1, \ldots, N_{n-1})$ is given by the number of non-negative integer solutions of the linear diophantine system

$$\left\{\begin{array}{ccccc} q_{1,1}x_n & + & \alpha x_1 & = & N_1 \\ \vdots & & \vdots & & \vdots \\ q_{n-1,1}x_n & + & \alpha x_{n-1} & = & N_{n-1}\,. \end{array}\right. \tag{3.16}$$

Proof of proposition 2.2. Recall that, choosing $\alpha > 0$, there is at least one index i for which $q_{i,1} > 0$. Otherwise

$$0 < \omega_n = \sum_{j=1}^{n-1} \frac{\nu_{1,j}}{\alpha}\omega_j = \sum_{j=1}^{n-1} \frac{q_{j,1}}{\alpha}\omega_j < 0\,.$$

Therefore in what follows we can assume (see remark (i) after proposition 2.2) the existence of $0 \leqslant r \leqslant n-2$ such that $\nu_{1,1}, \ldots, \nu_{1,r} < 0$; $\nu_{1,r+1}, \ldots, \nu_{1,n-1} > 0$. This implies, in particular, that $\nu_{1,n-1}$ is always taken positive. The case $\nu_{i,j} = 0$ for some (i, j) is disregarded because it reduces to the single independent equation $\alpha x_j = N_j$ and therefore does not affect the number of the solutions. The system (3.16) is equivalent to the following system of linear congruences:

$$\left.\begin{array}{ccc} q_{1,1}x_n & \equiv & N_1 \\ \vdots & & \vdots \\ q_{n-1,1}x_n & \equiv & N_{n-1} \end{array}\right\} \qquad (\operatorname{mod} \alpha) \tag{3.17}$$

which always admits at least the solution $x_n = p_n$. Let $D_N = \gcd(q_{1,1} \ldots, q_{n-1,1}, \alpha)$. Then the system (3.17) admits exactly D_n distinct solutions, denoted $y_n^h : h = 1, \ldots, D_n$ (see e.g. [Ca, section XVII.337]). These yield D_n distinct solutions of the system (3.16), in turn denoted by

$$y^h = (y_1^h, \ldots, y_n^h) \qquad h = 1, \ldots, D_n \tag{3.18}$$

whence the general solution of (3.16) under the form

$$(y_1^h, \dots, y_{n-1}^h, y_n^h) + (-kq_{1,1}, \dots, -kq_{n-1,1}, k\alpha) \qquad h = 1, \dots, D_n,\ k \in \mathbb{Z}. \tag{3.19}$$

Now take $D_n = 1$, and therefore let y_n be the unique solution of (3.17), which is obviously such that $0 \leqslant y_n < \alpha$. We now look for the conditions on k in (3.19) coming from the requirement of the non-negativity of the solutions.

(i) $y_n + k\alpha \geqslant 0 \Rightarrow k \geqslant 0$ (since $y_n < \alpha$).

(ii) $y_i - kq_{i,1} \geqslant 0$ for $r+1 \leqslant i \leqslant n-1$. Hence, since for these values of i we have $q_{i,1} > 0$, the same argument of the proof of proposition 2.1 applies and we have $k \leqslant L_1$, where

$$L_1 = \min_{r+1 \leqslant i \leqslant n-1} \left(\left[\frac{y_i}{q_{i,1}} \right] \right). \tag{3.20}$$

Remark also that $\left[y_i/q_{i,1}\right]$ fulfills, by the same argument of proposition 2.1, the estimate (3.9).

(iii) Consider now the equations for which $q_{j,1} < 0$, $j = 1, \dots, r$, namely $-|q_{j,1}|x_n + \alpha x_j = N_j$. Then the solutions are

$$(y_j, y_n) + (|q_{j,1}|, k\alpha). \tag{3.21}$$

If $N_j \geqslant 0$ then $y_j \geqslant 0$, which implies that the solution (3.21) is non-negative $\forall k \geqslant 0$. If $N_j < 0$, one must have $-|q_{j,1}|x_n \leqslant N_j$ which in turn yields $x_n \geqslant -N_j/|q_{j,1}|$. Therefore, if the solution y_n does not fulfill the lower bound $y_n \geqslant -N_j/|q_{j,1}|$, k has to be chosen in such a way that

$$y_n + k\alpha \geqslant -\frac{N_j}{|q_{j,1}|}. \tag{3.22}$$

We must therefore have

$$y_n + k\alpha \geqslant \max_{j \leqslant r} \left(0, \frac{N_j}{q_{j,1}} \right). \tag{3.23}$$

Therefore, on account of (i), we can conclude that $k \geqslant M_1$, where

$$M_1 = \max \left(0, \left(\frac{1}{\alpha} \left(\max_{j \leqslant r} \left(0, \frac{N_j}{q_{j,1}} \right) - y_n \right) \right) \right). \tag{3.24}$$

Putting together (ii) and (iii) it follows that in order to get non-negative solutions we must require

$$L_1 \leqslant k \leqslant M_1. \tag{3.25}$$

If $D_n > 1$ the same argument applies separately to any single one of the D_n distinct solutions $(y_1^h, \dots, y_n^h) : h = 1, \dots, D_n$. The assertion is therefore proved upon summation over $h = 1, \dots, D_n$. This concludes the proof of proposition 2.2.

We now proceed to the proof of proposition 2.3. To obtain it, we make use of a relatively recent result in number theory which we state under the form of a lemma.

Lemma 3.2. . Let $A \in M(m \times n)$ and $B \in M(m \times 1)$ be matrices with integer coefficients, and let $\mathrm{rank}(A) = r$. If the equation $Ax = 0$ does not have non-trivial non-negative solutions, then any non-negative integer solution $y = (y_1, \dots, y_n)$ of the equation $Ax = B$ fulfills the estimate

$$y_i \leqslant M \tag{3.26}$$

where M is the largest among the absolute values of the rank(r) minors of the matrix obtained augmenting A by B, namely completing A by the column matrix B.

Proof. See [BT, theorem 4].

As above, before starting the argument we recall that when there are s resonance relations $(1 < s \leqslant n-2)$ the 'principal quantum numbers' have the form

$$N_i^{(s)} = p_i\alpha + \sum_{j=1}^{s} q_{i,j}p_{n-s+j} \qquad i = 1, \ldots, n-s\,. \tag{3.27}$$

The multiplicity $M_n^s(\alpha, q_{i,j}|_{i=1,\ldots,n-s;j=1,\ldots,s}; N_1, \ldots, N_{n-s})$ of the eigenvalues is this time given by the number of non-negative solution of the linear diophantine system

$$\left\{\begin{array}{ccccccccc} q_{1,1}x_{n-s+1} & +\cdots+ & q_{1,s}x_n & + & \alpha x_1 & = & N_1 \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ q_{n-s,1}x_{n-s+1} & +\cdots+ & q_{n-s,s}x_n & + & \alpha x_{n-s} & = & N_{n-s}\,. \end{array}\right. \tag{3.28}$$

Proof of proposition 2.3. Following the same argument of the proof of proposition 2.1, consider the linear diophantine systems

$$\left\{\begin{array}{ccccc} q_{1,1}x_{n-s+1} & + & \alpha x_1 & = & R_1^H \\ \vdots & & \vdots & & \vdots \\ q_{n-s,1}x_{n-s+1} & + & \alpha x_{n-s} & = & R_{n-s}^H \end{array}\right. \tag{3.29}$$

parametrized by the non-negative integers $H = (h_{n-s+2}, \ldots, h_n) \in \mathbb{Z}_+^{s-1}$, where

$$R_i^H = N_i - \sum_{j=1}^{s} h_{n-s+j}q_{i,j} \qquad i = 1, \ldots, n-s\,. \tag{3.30}$$

Each system of the form (3.29) can be analysed in exactly the same way as the system (3.16). Recalling the definition of α and $q_{i,j}$ we can indeed always choose $\nu_{j,n-s+j} > 0$ whence $\alpha = \prod_{j=1}^{s} \nu_{j,n-s+j} > 0$. Moreover, since $\nu_{j,n-s+j} > 0$, for each single resonance relation, i.e. $\forall 1 \leqslant j \leqslant s$, there must be $i = i(j) : 1 \leqslant i \leqslant n-s$ such that $\nu_{j,i(j)} > 0$. This in turn implies that $\forall 1 \leqslant j \leqslant s\ \exists i = i(j)$ such that $q_{i,j} > 0$. In particular for $j = 1$ there is $i = i(1)$ such that $q_{i,1} > 0$. Let us now write down explicitly the conditions under which the systems (3.29) admit solutions (see e.g. [Ca section XVII. 334–6]). Let $e_1, \epsilon_1^H, \epsilon_2^H$ be the elementary divisors of the matrices

$$\begin{pmatrix} q_{1,1} \\ \vdots \\ q_{n-s,1} \end{pmatrix} \qquad \begin{pmatrix} q_{1,1} & R_1^H \\ \vdots & \vdots \\ q_{n-s,1} & R_{n-s}^H \end{pmatrix}.$$

Then the systems (3.29) admits solutions if and only if

$$\left\{\begin{array}{l} \gcd(\alpha, e_1) = \gcd(\alpha, \epsilon_1^H) \\ \epsilon_2^H \equiv 0 \pmod{\alpha}\,. \end{array}\right. \tag{3.31}$$

The existence of non-negative solutions requires, moreover,

$$R_i^H \geqslant 0 \qquad \forall i \quad \text{such that} \quad q_{i,1} \geqslant 0\,. \tag{3.32}$$

Therefore $\forall H \in \mathbb{Z}_+^{s-1}$ the number of non-negative solutions of (3.29) is given by $M_{n-s+1}^1(\alpha, q_{1,1}, \ldots, q_{n-s+1}; R_1^H, \ldots, R_{n-s}^H)$ if (3.31) and (3.32) hold, and by 0 otherwise. Therefore, by exactly the same argument as in the proof of proposition 2.2 we can write

$$M_n^s(\alpha, q_{i,j}|_{i=1,\ldots,n-s;j=1,\ldots,s}; N_1, \ldots, N_{n-s}) = \sum_{H=(h_{n-s+2},\ldots,h_n)\in\mathbb{Z}_+^s} M_{n-s+1}^1(\alpha, q_{1,1}, \ldots, q_{n-s,1}; R_1^H, \ldots, R_{n-s}^H) \tag{3.33}$$

where $M^{1}_{n-s+1}(\cdot)$ is extended to 0 if (3.31), (3.32) do not hold. Now remark that the system (3.28) fulfills the conditions of lemma 3.2. This is because the number of the non-negative solutions of the associated homogeneous system is the multiplicity of the eigenvalue λ_0 which is always simple. Therefore this last system cannot have non-negative solutions different from the trivial one. Therefore all non-negative solutions $(y_1, \ldots, y_n)$ of (3.28) must be such that

$$y_i \leqslant K \qquad i = 1, \ldots, n \tag{3.34}$$

where K is the maximum among the absolute values of the $n-s$-rank minors of the matrix

$$\begin{pmatrix} \alpha & 0 & \ldots & 0 & q_{1,1} & \ldots & q_{1,s} & N_1 \\ 0 & \alpha & 0 & 0 & q_{2,1} & \ldots & q_{2,s} & N_2 \\ \vdots & 0 & \ddots & \vdots & \vdots & & \vdots & \vdots \\ 0 & \ldots & 0 & \alpha & q_{n-s,1} & \ldots & q_{n-s,s} & N_{n-s} \end{pmatrix}. \tag{3.35}$$

We can therefore conclude that the multiple sum (3.33) is extended only to those indices $H = (h_{n-s+2}, \ldots, h_n)$ such that $0 \leqslant h_j \leqslant K$, $j = n-s+2, \ldots, n$, and this proves the assertion.

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